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Double Hyperbolic Splines on the Real Line

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INTRODUCTION

In this paper we extend the work in [2] on the existence and uniqueness of hyperbolic splines on R^1 to double hyperbolic and higher-order generalized splines, utilizing a new representation of a spline of interpolation, in terms of several of its unknown derivatives at mesh points, rather than just one. We also show how this approach may be applied to higher order polynomial splines on R^1. For the standard approach, see [1].

1. DOUBLE HYPERBOLIC AND QUINTIC SPLINES ON R^1

In this section we develop a technique, which we introduce by means of a specific example, for calculating higher-order generalized splines on R^1. We consider the linear differential operator

$$L = D(D-\alpha)(D-\beta), \qquad \alpha, \beta \neq 0, \quad \alpha \neq \beta,$$

and, without loss of generality, we take $0 < \alpha < \beta$. We call $S_\Delta(x)$ a *double hyperbolic spline* on R^1 for the uniform mesh $\Delta = \{x_j = jl: j = 0, \pm 1, \pm 2, ...\}$ if (i) on each mesh interval $[x_j, x_{j+1}]$, $S_\Delta(x)$ satisfies $L^*LS_\Delta(x) = 0$, and (ii) $S_\Delta(x)$ is $C^4(R^1)$. Furthermore, $S_\Delta(x)$ is a double hyperbolic spline of interpolation if, in addition, (iii) $S_\Delta(x_j) = y_j$, $j = 0, \pm 1, \pm 2, ...,$ for prescribed interpolation data $\{y_j\}$. We have $L^*L = -D^2(D^2 - \alpha^2)(D^2 - \beta^2)$, so on each mesh interval $[x_j, x_{j+1}]$,

$$S_\Delta(x) = c_1^j + c_2^j x + c_3^j \sinh \alpha x + c_4^j \cosh \alpha x + c_5^j \sinh \beta x + c_6^j \cosh \beta x.$$

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Instead of attempting to express S_Δ in terms of $S_\Delta^{(\mathrm{iv})}(x_j)$ and y_j only, as is standardly done for the quintic spline [1], we employ two unknowns $M_j = S_\Delta''(x_j)$ and $N_j = S_\Delta^{(\mathrm{iv})}(x_j)$ in our representation of the double hyperbolic (and quintic) spline on R^1. We get the following equations for the c_i^j, $i = 1,\ldots, 6, j = 0, \pm 1, \pm 2,\ldots$, in terms of $\{y_j\}, \{M_j\}, \{N_j\}$.

$$
\begin{aligned}
&c_1^j + c_2^j x_j + c_3^j \sinh \alpha x_j + c_4^j \cosh \alpha x_j + c_5^j \sinh \beta x_j + c_6^j \cosh \beta x_j = y_j\,,\\
&c_1^j + c_2^j x_{j+1} + c_3^j \sinh \alpha x_{j+1} + c_4^j \cosh \alpha x_{j+1} + c_5^j \sinh \beta x_{j+1}\\
&\qquad\qquad + c_6^j \cosh \beta x_{j+1} = y_{j+1}\,,\\
&\alpha^2 c_3^j \sinh \alpha x_j + \alpha^2 c_4^j \cosh \alpha x_j + \beta^2 c_5^j \sinh \beta x_j + \beta^2 c_6^j \cosh \beta x_j = M_j\,,\\
&\alpha^2 c_3^j \sinh \alpha x_{j+1} + \alpha^2 c_4^j \cosh \alpha x_{j+1} + \beta^2 c_5^j \sinh x_{j+1}\\
&\qquad\qquad + \beta^2 c_6^j \cosh \beta x_{j+1} = M_{j+1},\\
&\alpha^4 c_3^j \sinh \alpha x_j + \alpha^4 c_4^j \cosh \alpha x_j + \beta^4 c_5^j \sinh \beta x_j + \beta^4 c_6^j \cosh \beta x_j = N_j\,,\\
&\alpha^4 c_3^j \sinh \alpha x_{j+1} + \alpha^4 c_4^j \cosh \alpha x_{j+1} + \beta^4 c_5^j \sinh \beta x_{j+1}\\
&\qquad\qquad + \beta^4 c_6^j \cosh \beta x_{j+1} = N_{j+1}\,.
\end{aligned}
\tag{1.1}
$$

Solution of these results in the following representation for S_Δ on $[x_j\,, x_{j+1}]$.

$$
\begin{aligned}
S_\Delta(x) = {} & y_j \left(\frac{x_{j+1} - x}{l}\right) + y_{j+1} \left(\frac{x - x_j}{l}\right) + (\alpha^2\beta^2(\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l)^{-1}\\
& \cdot M_j \Big[-(\beta^4 - \alpha^4) \sinh \alpha l \sinh \beta l \left(\frac{x_{j+1} - x}{l}\right) + \beta^4 \sinh \beta l\\
& \cdot \sinh \alpha(x_{j+1} - x) - \alpha^4 \sinh \alpha l \sinh \beta(x_{j+1} - x)\Big]\\
& + M_{j+1} \Big[-(\beta^4 - \alpha^4) \sinh \alpha l \sinh \beta l \left(\frac{x - x_j}{l}\right) + \beta^4 \sinh \beta l\\
& \cdot \sinh \alpha(x - x_j) - \alpha^4 \sinh \alpha l \sinh \beta(x - x_j)\Big]\\
& + N_j \Big[(\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l \left(\frac{x_{j+1} - x}{l}\right) - \beta^2 \sinh \beta l\\
& \cdot \sinh \alpha(x_{j+1} - x) + \alpha^2 \sinh \alpha l \sinh \beta(x_{j+1} - x)\Big]\\
& + N_{j+1} \Big[(\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l \left(\frac{x - x_j}{l}\right) - \beta^2 \sinh \beta l\\
& \cdot \sinh \alpha(x - x_j) + \alpha^2 \sinh \alpha l \sinh \beta(x - x_j)\Big],
\end{aligned}
\tag{1.2}
$$

for all j. By definition, $S_\Delta(x_j) = y_j$ and S_Δ, S_Δ'', and $S_\Delta^{(\mathrm{iv})}$ are continuous at x_j. The two requirements that S_Δ' and S_Δ''' be continuous at x_j lead to two equations in the two unknowns M_j, N_j for each j. In matrix form these become

$$\begin{bmatrix} & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdots & 0 & c_1 & c_2 & c_3 & c_4 & c_1 & c_2 & 0 & \cdot & & & \\ \cdots & 0 & d_1 & d_2 & d_3 & d_4 & d_1 & d_2 & 0 & \cdots & & & \\ & & \cdots & 0 & c_1 & c_2 & c_3 & c_4 & c_1 & c_2 & 0 & \cdots & \\ & & \cdots & 0 & d_1 & d_2 & d_3 & d_4 & d_1 & d_2 & 0 & \cdots & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \end{bmatrix} \begin{bmatrix} \vdots \\ M_{-1} \\ N_{-1} \\ M_0 \\ N_0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a_{-1} \\ 0 \\ a_0 \\ 0 \\ \vdots \end{bmatrix}, \tag{1.3}$$

where

$$\begin{aligned}
c_1 &= [(\beta^4 - \alpha^4) \sinh \alpha l \sinh \beta l - \alpha l \beta^4 \sinh \beta l + \alpha^4 \beta l \sinh \alpha l]/\mathscr{C}, \\
c_3 &= -2[(\beta^4 - \alpha^4) \sinh \alpha l \sinh \beta l - \alpha l \beta^4 \sinh \beta l \cosh \alpha l \\
&\qquad + \alpha^4 \beta l \sinh \alpha l \cosh \beta l]/\mathscr{C}, \\
c_2 &= [-(\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l + \alpha l \beta^2 \sinh \beta l - \alpha^2 \beta l \sinh \alpha l]/\mathscr{C}, \\
c_4 &= -2[-(\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l + \alpha l \beta^2 \sinh \beta l \cosh \alpha l \\
&\qquad - \alpha^2 \beta l \sinh \alpha l \cosh \beta l]/\mathscr{C}, \\
\mathscr{C} &= \alpha^2 \beta^2 l^2 (\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l, \\
d_1 &= [-\alpha l \beta^2 \sinh \beta l + \alpha^2 \beta l \sinh \alpha l]/\mathscr{D}, \\
d_3 &= -2[-\alpha l \beta^2 \sinh \beta l \cosh \alpha l + \alpha^2 \beta l \sinh \alpha l \cosh \beta l]/\mathscr{D}, \\
d_2 &= [\alpha l \sinh \beta l - \beta l \sinh \alpha l]/\mathscr{D}, \\
d_4 &= -2[\alpha l \sinh \beta l \cosh \alpha l - \beta l \sinh \alpha l \cosh \beta l]/\mathscr{D}, \\
\mathscr{D} &= (\beta^2 - \alpha^2) \sinh \alpha l \sinh \beta l,
\end{aligned}$$

and

$$a_j = (1/l^2)(y_{j+1} - 2y_j + y_{j-1}).$$

Representation (1.2) defines a unique double hyperbolic spline of interpolation provided only that Eqs. (1.3) define uniquely the M_j's and N_j's. Instead of attempting to invert the doubly infinite block Toeplitz matrix in (1.3), we note that (1.3) is equivalent to the following pair of systems of equations

$$\begin{aligned} C_1 \mathbf{m} + C_2 \mathbf{n} &= \mathbf{a}, \\ D_1 \mathbf{m} + D_2 \mathbf{n} &= \mathbf{0}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} C_1 &= [\cdots\ 0\ c_1\ c_3\ c_1\ 0\ \cdots],^1 \\ C_2 &= [\cdots\ 0\ c_2\ c_4\ c_2\ 0\ \cdots], \\ D_1 &= [\cdots\ 0\ d_1\ d_3\ d_1\ 0\ \cdots], \\ D_2 &= [\cdots\ 0\ d_2\ d_4\ d_2\ 0\ \cdots], \end{aligned}$$

and

$$\mathbf{m} = \begin{bmatrix} \vdots \\ M_{-1} \\ M_0 \\ M_1 \\ \vdots \end{bmatrix}, \qquad \mathbf{n} = \begin{bmatrix} \vdots \\ N_{-1} \\ N_0 \\ N_1 \\ \vdots \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} \vdots \\ a_{-1} \\ a_0 \\ a_1 \\ \vdots \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

We attempt to solve this system for the unknowns $\mathbf{m}$, $\mathbf{n}$ by means of substitution. We recall an important theorem on the invertibility of doubly infinite Toeplitz matrices. Consider the Toeplitz matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_0 & c_{-1} & c_{-2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_1 & c_0 & c_{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_2 & c_1 & c_0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

and define $\phi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{int}$. If the c_n are such that ϕ is a bounded function, then we have

THEOREM 1.1. *T_ϕ is invertible if $1/\phi$ is essentially bounded. If T_ϕ^{-1} exists, it satisfies $T_\phi^{-1} = T_{1/\phi}$ where $T_{1/\phi}$ is the Toeplitz matrix defined by the sequence of Fourier coefficients of* $1/\phi$ [5].

First we note that

THEOREM 1.2. *D_2 is invertible for all $\beta > \alpha > 0$.*

For the proof of this we require the following.

LEMMA 1.3. *If $y > x$ then* (i) $y \sinh x \cosh y > x \sinh y \cosh x$ *and* (ii) $x \sinh y > y \sinh x$, *both for* $x, y > 0$.

Proof. Consider the function $f(t) = (t \cosh t)/\sinh t$ for $t > 0$. Then $f'(t) = (\sinh t \cosh t - t)/\sinh^2 t$. Now let $g(t) = \sinh t \cosh t - t$. Then

[1] We use this notation to denote a doubly infinite Toeplitz matrix in which a row may be obtained from the preceding one by shifting all elements one column to the right.

$g(0) = 0$ and $g'(t) = 2 \sinh^2 t > 0$ for $t > 0$. Since $g(t) = g(0) + \int_0^t g'(s)\, ds$ $g(t) > 0$ for $t > 0$. Thus $f'(t) > 0$ for $t > 0$ and so f is monotonically strictly increasing for $t > 0$. Hence $f(y) > f(x)$ for $y > x$, which implies $y \sinh x \cosh y > x \sinh y \cosh x$. (ii) is proved analogously.

Proof of Theorem 1.2. By Theorem 1.1, D_2 is invertible iff $1/(d_4 + 2d_2 \cos\theta)$ is essentially bounded (for $0 < \alpha < \beta$) iff $d_4 + 2d_2 \cos\theta \neq 0$ for all θ iff $| d_4 | > 2 | d_2 |$; and this can be seen as follows. Lemma 1.3 implies $| d_4 | = d_4$ and $| d_2 | = d_2$. So $| d_4 | > 2 | d_2 |$ iff

$$\beta l \sinh \alpha l \cosh \alpha l - \alpha l \sinh \beta l \cosh \alpha l > \alpha l \sinh \beta l - \beta l \sinh \alpha l$$

iff

$$\beta l(\cosh \beta l + 1)/\sinh \beta l > \alpha l(\cosh \alpha l + 1)/\sinh \alpha l \qquad \text{for} \quad 0 < \alpha < \beta. \tag{1.5}$$

Now let $f(t) = t(\cosh t + 1)/\sinh t$. So $f'(t) = (\cosh t + 1)(\sinh t - t)/\sinh^2 t$, and clearly $f'(t) > 0$ for $t > 0$. Thus f is monotonically strictly increasing for $t > 0$ which yields the necessary inequality (1.5) for the invertibility of D_2.

The second system of equations in (1.4) yields

$$\mathbf{n} = -D_2^{-1} D_1 \mathbf{m}, \tag{1.6}$$

and substitution into the first system gives

$$[C_1 - C_2 D_2^{-1} D_1]\mathbf{m} = \mathbf{a}. \tag{1.7}$$

Before answering the question of the invertibility of the matrix in (1.7), we make a diversion from the case of the double hyperbolic spline to the associated case of the quintic spline ($\alpha = \beta = 0$) in which we show how to invert the matrix, similar to the one in (1.7), which arises; the procedure carries over to the matrix in (1.7) whenever it is invertible.

Utilizing the same technique as earlier in this section, we find for the operator $L = D^3$ on $[x_j, x_{j+1}]$,

$$\begin{aligned} S_\Delta(x) = {} & y_j \left(\frac{x_{j+1} - x}{l}\right) + y_{j+1} \left(\frac{x - x_j}{l}\right) + M_j \left[-\frac{l}{6}(x_{j+1} - x)\right. \\ & \left. + \frac{1}{6l}(x_{j+1} - x)^3\right] + M_{j+1} \left[-\frac{l}{6}(x - x_j) + \frac{1}{6l}(x - x_j)^3\right] \\ & + N_j \left[\frac{7l^3}{360}(x_{j+1} - x) - \frac{l}{36}(x_{j+1} - x)^3 + \frac{1}{120l}(x_{j+1} - x)^5\right] \\ & + N_{j+1} \left[\frac{7l^3}{360}(x - x_j) - \frac{l}{36}(x - x_j)^3 + \frac{1}{120l}(x - x_j)^5\right], \end{aligned} \tag{1.8}$$

and there result, after some straightforward computations,

$$\begin{aligned}(1/6)[\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots]\mathbf{m} - (l^2/360)[\cdots\ 0\ 7\ 16\ 7\ 0\ \cdots]\mathbf{n} &= \mathbf{a},\\ [\cdots\ 0\ -1\ 2\ -1\ 0\ \cdots]\mathbf{m} + (l^2/6)[\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots]\mathbf{n} &= \mathbf{0}\end{aligned} \tag{1.9}$$

as the equations of continuity of the first and third derivatives of S_Δ at x_j, $j = 0, \pm 1, \pm 2, \ldots$.

Using the same names, C_1, C_2, D_1, D_2, for the matrices in (1.9) corresponding to those in (1.4), we observe the following important fact in both cases. Since D_2 is always of the form $K_1[\cdots\ 0\ 1\ 2\eta\ 1\ 0\ \cdots]$, $\eta > 1$, and C_2 is of the form $K_2[\cdots\ 0\ 1\ 2\nu\ 1\ 0\ \cdots]$, although we claim nothing about the size of ν, we have

LEMMA 1.4. $C_2 D_2^{-1} = D_2^{-1} C_2$.

Proof. $D_2^{-1} = (1/2(\eta^2 - 1)^{1/2})[\cdots\ \mu^2\ \mu\ 1\ \mu\ \mu^2\ \cdots]$ (an application of Theorem 1.1; see [2]), so

$$\begin{aligned}&[\cdots\ 0\ 1\ 2\nu\ 1\ 0\ \cdots][\cdots\ \mu^2\ \mu\ 1\ \mu\ \mu^2\ \cdots]\\ &\quad = [\cdots\ \mu^2(2\nu\mu + 1 + \mu^2)\ \mu(2\nu\mu + 1 + \mu^2)\ (2\nu\mu + 1 + \mu^2)\\ &\qquad\quad 2(\nu + \mu)\ (2\nu\mu + 1 + \mu^2)\ \mu(2\nu\mu + 1 + \mu^2)\ \mu^2(2\nu\mu + 1 + \mu^2)\ \cdots]\\ &\quad = [\cdots\ \mu^2\ \mu\ 1\ \mu\ \mu^2\ \cdots][\cdots\ 0\ 1\ 2\nu\ 1\ 0\ \cdots].\end{aligned}$$

Therefore, Eq. (1.7) becomes

$$[C_1 - D_2^{-1} C_2 D_1]\mathbf{m} = \mathbf{a}. \tag{1.10}$$

Now, because D_2 is invertible, $D_2[C_1 - D_2^{-1} C_2 D_1] = D_2 C_1 - C_2 D_1$ is invertible iff $[C_1 - C_2 D_2^{-1} D_1]$ is. So we left multiply Eq. (1.10) by D_2 and attempt to solve

$$[D_2 C_1 - C_2 D_1]\mathbf{m} = D_2 \mathbf{a}. \tag{1.11}$$

Whenever $[D_2 C_1 - C_2 D_1]$ is invertible,

$$\mathbf{m} = [D_2 C_1 - C_2 D_1]^{-1} D_2 \mathbf{a}, \tag{1.12}$$

and from (1.6)

$$\mathbf{n} = -D_2^{-1} D_1 [D_2 C_1 - C_2 D_1]^{-1} D_2 \mathbf{a}. \tag{1.13}$$

We will show momentarily that $[D_2 C_1 - C_2 D_1]$, when it exists, is of the form $K_3[\cdots\ \lambda^2\ \lambda\ 1\ \lambda\ \lambda^2\ \cdots] - K_4[\cdots\ \zeta^2\ \zeta\ 1\ \zeta\ \zeta^2\ \cdots]$, with $-1 < \lambda$, $\zeta < 0$; and, therefore, by the same argument used to establish Lemma 1.4, it is clear that

D_2 commutes with $[D_2C_1 - C_2D_1]^{-1}$, and D_1 commutes with D_2^{-1}. Hence (1.13) becomes

$$\mathbf{n} = -D_1D_2^{-1}D_2[D_2C_1 - C_2D_1]^{-1}\,\mathbf{a} = -D_1[D_2C_1 - C_2D_1]^{-1}\,\mathbf{a}. \quad (1.14)$$

We see from Eqs. (1.12) and (1.14) that it is only necessary to invert one Toeplitz matrix in order to find $\mathbf{m}$ and $\mathbf{n}$ and thereby evaluate the quintic or double hyperbolic splines on R^1. The existence of these splines, provided $[D_2C_1 - C_2D_1]$ is invertible, becomes a question (as was the case in [2]) of the choice of the interpolation data $\{y_j\}$ in such a manner that the necessary sums represented in (1.12) and (1.14) for $\mathbf{m}$ and $\mathbf{n}$ converge.

Now we come to the crux of the matter, the invertibility of $[D_2C_1 - C_2D_1]$. In the quintic case (1.9), (1.11) becomes

$$\begin{aligned}&\{(l^2/36)[\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots][\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots]\\ &\quad + (l^2/360)[\cdots\ 0\ 7\ 16\ 7\ 0\ \cdots][\cdots\ 0\ -1\ 2\ -1\ 0\ \cdots]\}\mathbf{m}\\ &\quad = (l^2/6)[\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots]\mathbf{a}, \end{aligned} \quad (1.15)$$

or

$$(l^2/120)[\cdots\ 0\ 1\ 26\ 66\ 26\ 1\ 0\ \cdots]\mathbf{m} = (l^2/6)[\cdots\ 0\ 1\ 4\ 1\ 0\ \cdots]\mathbf{a}, \quad (1.16)$$

and so we define the function ϕ as

$$\begin{aligned}\phi(\theta) &= (l^2/120)(66 + 52\cos\theta + 2\cos 2\theta)\\ &= (l^2/120)(66 + 52\cos\theta + 2(2\cos^2\theta - 1)) \\ &= (l^2/30)(\cos^2\theta + 13\cos\theta + 16).\end{aligned} \quad (1.17)$$

Again, to invert the matrix on the left-hand side of (1.16) we must calculate the Fourier coefficients of $1/\phi(\theta)$. First note that $\phi(\theta) \neq 0$ for all θ since

$$\phi(\theta) = \frac{l^2}{30}\left(\cos\theta + \frac{13 + (105)^{1/2}}{2}\right)\left(\cos\theta + \frac{13 - (105)^{1/2}}{2}\right),$$

and $((13 \pm (105)^{1/2})/2) > 1$. Now, letting a_n be the Fourier coefficients of $1/\phi$, we find by a computation which we spare the reader, involving the evaluation of coefficients by residues, that

$$a_{-n} = a_n = \frac{-1}{(105)^{1/2}}\left[\frac{(-r + (r^2 - 1)^{1/2})^n}{(r^2 - 1)^{1/2}} - \frac{(-s + (s^2 - 1)^{1/2})^n}{(s^2 - 1)^{1/2}}\right],$$

where $r = ((13 + (105)^{1/2})/2)$ and $s = ((13 - (105)^{1/2})/2)$, for $n = 0, 1, 2,\ldots$. So $[D_2C_1 - C_2D_1]^{-1}$ is

$$\frac{-30}{l^2(105)^{1/2}}\left\{\frac{1}{(r^2 - 1)^{1/2}}[\cdots\ \lambda^2\ \lambda\ 1\ \lambda\ \lambda^2\ \cdots] - \frac{1}{(s^2 - 1)^{1/2}}[\cdots\ \zeta^2\ \zeta\ 1\ \zeta\ \zeta^2\ \cdots]\right\}, \quad (1.18)$$

where $\lambda = -r + (r^2 - 1)^{1/2}$ and $\zeta = -s + (s^2 - 1)^{1/2}$. Equations (1.12) and (1.14) then become

$$\mathbf{m} = \frac{-5}{(105)^{1/2}} \left\{ \frac{1}{(r^2-1)^{1/2}} [\cdots\ \lambda^2(1+4\lambda+\lambda^2)\ \ \lambda(1+4\lambda+\lambda^2)\ \ (1+4\lambda+\lambda^2)\right.$$
$$2(2+\lambda)\ \ (1+4+\lambda^2)\ \ \lambda(1+4\lambda+\lambda^2)\ \ \lambda^2(1+4\lambda+\lambda^2)\ \cdots]$$
$$- \frac{1}{(s^2-1)^{1/2}} [\cdots\ \zeta^2(1+4\zeta+\zeta^2)\ \ \zeta(1+4\zeta+\zeta^2)\ \ (1+4\zeta+\zeta^2)$$
$$\left. 2(2+\zeta)\ \ (1+4\zeta+\zeta^2)\ \ \zeta(1+4\zeta+\zeta^2)\ \ \zeta^2(1+4\zeta+\zeta^2)\ \cdots] \right\} \mathbf{a},$$

and

$$\mathbf{n} = \frac{-30}{l^2(105)^{1/2}} \left\{ \frac{(1-\lambda)}{(r^2-1)^{1/2}} [\cdots\ \lambda^2(\lambda-1)\ \ \lambda(\lambda-1)\ \ (\lambda-1)\ \ 2\ \ (\lambda-1)\ \ \lambda(\lambda-1)\right.$$
$$\lambda^2(\lambda-1)\ \cdots]$$
$$- \frac{(1-\zeta)}{(s^2-1)^{1/2}} [\cdots\ \zeta^2(\zeta-1)\ \ \zeta(\zeta-1)\ \ (\zeta-1)\ \ 2\ \ (\zeta-1)\ \ \zeta(\zeta-1)$$
$$\left. \zeta^2(\zeta-1)\ \cdots] \right\} \mathbf{a}. \tag{1.19}$$

We return now to the question of the invertibility of the matrix in (1.11). First, we will show that the entries in the matrices in (1.4) converge to the corresponding ones in (1.9) in the limit as $\alpha, \beta \to 0$ for a suitable choice of $\alpha < \beta$. We have

LEMMA 1.5. *Let $\alpha = a\beta^n$ for $n \geqslant 2$, $0 < a < 1$, and let $\beta \to 0$. Then, for c_i, d_i, $i = 1,\ldots, 4$, defined following* (1.3), *we have*

$$\lim_{\beta\to 0} c_1 = 1/6, \quad \lim_{\beta\to 0} c_3 = 2/3, \quad \lim_{\beta\to 0} c_2 = -7l^2/360, \quad \lim_{\beta\to 0} c_4 = -16l^2/360,$$
$$\lim_{\beta\to 0} d_1 = -1, \quad \lim_{\beta\to 0} d_3 = 2, \quad \lim_{\beta\to 0} d_2 = l^2/6, \quad \lim_{\beta\to 0} d_4 = 2l^2/3.$$

Proof. For small β, from the definition of c_1 and the Taylor expansion of $\sinh x$ for small x, we have

$$c_1 = (a^2\beta^{2n}\beta^2 l^2(\beta^2 - a^2\beta^{2n})(a\beta^n l + \cdots)(\beta l + \cdots))^{-1}$$
$$\cdot \left[(\beta^4 - a^4\beta^{4n}) \left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right)\right.$$
$$\left. - a\beta^n l\beta^4 \left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right) + a^4\beta^{4n}\beta l \left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\right]$$

$$= (a^3\beta^{3n+5}l^4 + \cdots)^{-1}\Big[\beta^4\left(\frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right)$$

$$- a^4\beta^{4n}\left(\frac{\beta^3 l^3}{3!} + \cdots\right)\left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\Big]$$

$$= (a^3\beta^{3n+5}l^4 + \cdots)^{-1}\left(\frac{a^3\beta^{3n+5}l^4}{3!} + \cdots\right)$$

$$= \frac{1 + o(\beta)}{6 + o(\beta)} \to \frac{1}{6}, \qquad \text{as} \qquad \beta \to 0,$$

where $\cdots$ indicates lower order terms. The other terms are handled similarly. A similar argument holds if $n = 1$, i.e., if $\alpha = a\beta$, $0 < a < 1$, as $\beta \to 0$. In that case, the dominant term in the denominator has a factor of $\beta^2 - a^2\beta^{2n} = \beta^2(1 - a^2)$, and in the numerator, terms involving β^{5n+3} are of the same order (β^8) as β^{3n+5} and combine to give an identical factor of $(1 - a^2)$. So we have the required convergence for $\alpha = a\beta^n$, $0 < a < 1$, $n \geqslant 1$, as $\beta \to 0$.

It follows from the above that the entries in the matrix $D_2C_1 - C_2D_1$ in Eq. (1.11) converge to those in (1.15). Since there is only a finite number of nonzero entries in any row (and all the rows are identical except for a shift), we may conclude that, for sufficiently small β, the matrix in (1.11) is invertible by virtue of the following argument.

Let E be the Banach space of all bounded doubly infinite sequences of reals with sup norm; i.e., $E = \{x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) : \| x \| = \sup_{-\infty<i<\infty} | x_i | < \infty\}$. Then the matrix $D_2C_1 - C_2D_1$ of (1.15) or (1.16) is a bounded linear operator, in the induced operator norm, mapping E to E. Call this operator T. It is clear that T is linear and that it is bounded follows from the fact that

$$\| Tx \| = \sup_{-\infty<i<\infty} |(Tx)_i| = \sup_{-\infty<i<\infty} \left| e_0 x_i + \sum_{n=1}^{2} e_n(x_{i+n} + x_{i-n}) \right|$$

$$\leqslant \| x \| \left(| e_0 | + 2 \sum_{n=1}^{2} | e_n | \right) = \| x \| B,$$

where $T = [\cdots 0\, e_2\, e_1\, e_0\, e_1\, e_2\, 0 \cdots]$. Similarly, $\tilde{T}$ is the bounded linear operator defined by $D_2C_1 - C_2D_1$ of (1.11). We have the well-known [3]

THEOREM 1.6. *If T is a bounded linear operator from E to E having a bounded linear inverse T^{-1}, then any bounded linear operator $\tilde{T}$ satisfying $\| T - \tilde{T} \| < \| T^{-1} \|^{-1}$ has a bounded linear inverse $\tilde{T}^{-1}$ satisfying*

$$\| \tilde{T}^{-1} - T^{-1} \| \leqslant \| T^{-1} \|^2 \| T - \tilde{T} \| / (1 - \| T - \tilde{T} \| \cdot \| T^{-1} \|).$$

Since, by the choice of α, β we make $\| T - \tilde{T} \|$ arbitrarily small, it follows that the matrix in (1.11) is invertible for such values of α, β, and that $\tilde{T}^{-1}$ is close to T^{-1} as measured in the induced operator norm, which is, in fact, the row–max norm. (For any bounded linear operator on E, not just a symmetric Toeplitz matrix, the bound B on the norm of the operator, obtained in a manner similar to that on the preceding page, can be reached by an appropriate choice of the sequence x, having values ± 1; therefore, the norm of such an operator is, in fact, $B = \sup_{-\infty<i<\infty} \sum_{j=-\infty}^{\infty} | e_{ij} |$, the sup of the sums of the absolute values of the entries on each row.)

It is also true that the coefficients of the unknowns in representation (1.2) converge to the corresponding ones in representation (1.8) as $\alpha, \beta \to 0$. We see this in

LEMMA 1.7. *The coefficients of* M_j, M_{j+1}, N_j, N_{j+1} *in* (1.2) *converge, respectively, to those in* (1.8) *if* $\alpha = a\beta^n$, $n \geqslant 2$, $0 < a < 1$, *and* $\beta \to 0$.

Proof. Consider the coefficient of M_j in (1.2), call it z. Then for small β,

$$\begin{aligned}
z &= (a^2\beta^{2n}\beta^2(\beta^2 - a^2\beta^{2n})(a\beta^n l + \cdots)(\beta l + \cdots))^{-1} \\
&\quad \cdot \Big[-(\beta^4 - a^4\beta^{4n}) \left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right)\left(\frac{x_{j+1} - x}{l}\right) \\
&\quad + \beta^4 \left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right)\left(a\beta^n(x_{j+1} - x) + \frac{a^3\beta^{3n}(x_{j+1} - x)^3}{3!} + \cdots\right) \\
&\quad - a^4\beta^{4n} \left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\left(\beta(x_{j+1} - x) + \frac{\beta^3(x_{j+1} - x)^3}{3!} + \cdots\right)\Big] \\
&= (a^3\beta^{3n+5}l^2 + \cdots)^{-1} \Big[\beta^4 \left(\beta l + \frac{\beta^3 l^3}{3!} + \cdots\right)\Big(\frac{a^3\beta^{3n}}{3!} ((x_{j+1} - x)^3 \\
&\quad - l^2(x_{j+1} - x)) + \frac{a^5\beta^{5n}}{5!} ((x_{j+1} - x)^5 - l^4(x_{j+1} - x)) + \cdots\Big) \\
&\quad + a^4\beta^{4n} \left(a\beta^n l + \frac{a^3\beta^{3n}l^3}{3!} + \cdots\right)\Big(\frac{\beta^3}{3!} (l^2(x_{j+1} - x) - (x_{j+1} - x)^3) \\
&\quad + \frac{\beta^5}{5!} (l^4(x_{j+1} - x) - (x_{j+1} - x)^5) + \cdots\Big)\Big] \\
&= (a^3\beta^{3n+5}l^2 + \cdots)^{-1} \left(\frac{a^3\beta^{3n+5}l}{3!} ((x_{j+1} - x)^3 - l^2(x_{j+1} - x)) + \cdots\right) \\
&\to -\frac{l}{6} (x_{j+1} - x) + \frac{1}{6l} (x_{j+1} - x)^3, \qquad \text{as} \qquad \beta \to 0.
\end{aligned}$$

Again, the other coefficients are handled similarly, and, in this case too, the results hold even for $n = 1$.

Lemmas 1.5 and 1.7 and Theorem 1.6 allow us to establish the following.

THEOREM 1.8. *For $\alpha = a\beta^n$, $0 < a < 1$, $n \geqslant 1$, the double hyperbolic spline on R^1 converges to the quintic spline on R^1 in the limit as $\beta \to 0$.*

Proof. Equations (1.11) and (1.15) take the form

$$\tilde{T}\tilde{\mathbf{m}} = \tilde{D}_2\mathbf{a} \qquad \text{and} \qquad T\mathbf{m} = D_2\mathbf{a},$$

respectively. Therefore,

$$\begin{aligned}\| \tilde{\mathbf{m}} - \mathbf{m} \| &= \|(\tilde{T}^{-1}\tilde{D}_2 - T^{-1}D_2)\mathbf{a}\| \\ &= \|(\tilde{T}^{-1}\tilde{D}_2 - \tilde{T}^{-1}D_2 + \tilde{T}^{-1}D_2 - T^{-1}D_2)\mathbf{a}\| \\ &\leqslant (\|\tilde{T}^{-1}\| \cdot \|\tilde{D}_2 - D_2\| + \|D_2\| \cdot \|\tilde{T}^{-1} - T^{-1}\|)\|\mathbf{a}\|,\end{aligned}$$

which shows that $\tilde{\mathbf{m}}$ converges to $\mathbf{m}$, and a similar argument shows that $\tilde{\mathbf{n}}$ converges to $\mathbf{n}$, as $\beta \to 0$. Thus, the double hyperbolic spline defined by (1.2) must converge to the quintic spline defined by (1.8) as $\beta \to 0$.

For the double hyperbolic spline, the invertibility of $[D_2C_1 - C_2D_1]$ is equivalent to the essential boundedness of $1/\phi$, where

$$\begin{aligned}\phi(\theta) = (d_4c_3 - c_4d_3) + 2((d_4c_1 - c_4d_1) + (d_2c_3 - c_2d_3))& \\ \cdot \cos\theta + 4(d_2c_1 - c_2d_1)\cos^2\theta.&\end{aligned}$$

It is, therefore, equivalent to the statement "the equation

$$\begin{aligned}&[\alpha^3 l(\sinh\beta l - \beta l) - \beta^3 l(\sinh\alpha l - \alpha l)]\,x^2 \\ &-[\alpha^3 l(\sinh\beta l(1 + \cosh\alpha l) - \beta l(\cosh\alpha l + \cosh\beta l)) \\ &-\beta^3 l(\sinh\alpha l(1 + \cosh\beta l) - \alpha l(\cosh\alpha l + \cosh\beta l))]x \\ &+ [\alpha^3 l\cosh\alpha l(\sinh\beta l - \beta l\cosh\beta l) - \beta^3 l\cosh\beta l(\sinh\alpha l - \alpha l\cosh\alpha l)] = 0\end{aligned} \tag{1.20}$$

has no real roots with magnitude less than or equal to 1." A simple computer program devised to test this for arbitrary $0 < \alpha < \beta$, $l > 0$, lends credence to the fact that $[D_2C_1 - C_2D_1]$ is, in fact, invertible for all such α, β, although a rigorous proof escapes the author.

2. HIGHER-ORDER GENERALIZED SPLINES ON R^1

Having seen how the procedure of using several unknown spline derivatives in the representation of a spline works for the double hyperbolic and quintic splines on R^1 in the previous section, we now investigate the technique for higher-order cases. We consider linear differential operators of the form

$$L = D(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n), \tag{2.1}$$

for $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. A generalized spline satisfying $L^*LS_\Delta = 0$ on the mesh intervals can be written

$$S_\Delta(x) = c_0{}^j + c_1{}^j x + \sum_{i=1}^{n} (a_i{}^j \sinh \alpha_i x + b_i{}^j \cosh \alpha_i x) \tag{2.2}$$

on $[x_j, x_{j+1}]$. Using the known interpolation data y_j, y_{j+1} and the $2n$ unknowns, $M_j{}^1, M_{j+1}^1, M_j{}^2, M_{j+1}^2, \ldots, M_j{}^n, M_{j\pm 1}^n$, representing, respectively, $S_\Delta''(x_j), S_\Delta''(x_{j+1}), S_\Delta^{(\mathrm{iv})}(x_j), S_\Delta^{(\mathrm{iv})}(x_{j+1}), \ldots, S_\Delta^{(2n)}(x_j), S_\Delta^{(2n)}(x_{j+1})$, we have a system of $2n + 2$ equations for the $2n + 2$ unknown coefficients $c_0{}^j$, $c_1{}^j$, $\{a_i{}^j\}_{i=1}^n$, $\{b_i{}^j\}_{i=1}^n$:

$$\begin{bmatrix} 1 & x_j & \sinh\alpha_1 x_j & \cosh\alpha_1 x_j & \cdots & \sinh\alpha_n x_j & \cosh\alpha_n x_j \\ 1 & x_{j+1} & \sinh\alpha_1 x_{j+1} & \cosh\alpha_1 x_{j+1} & \cdots & \sinh\alpha_n x_{j+1} & \cosh\alpha_n x_{j+1} \\ 0 & 0 & \alpha_1{}^2\sinh\alpha_1 x_j & \alpha_1{}^2\cosh\alpha_1 x_j & \cdots & \alpha_n{}^2\sinh\alpha_n x_j & \alpha_n{}^2\cosh\alpha_n x_j \\ 0 & 0 & \alpha_1{}^2\sinh\alpha_1 x_{j+1} & \alpha_1{}^2\cosh\alpha_1 x_{j+1} & \cdots & \alpha_n{}^2\sinh\alpha_n x_{j+1} & \alpha_n{}^2\cosh\alpha_n x_{j+1} \\ \vdots & & & & & & \vdots \\ 0 & 0 & \alpha_1^{2n}\sinh\alpha_1 x_j & \alpha_1^{2n}\cosh\alpha_1 x_j & \cdots & \alpha_n^{2n}\sinh\alpha_n x_j & \alpha_n^{2n}\cosh\alpha_n x_j \\ 0 & 0 & \alpha_1^{2n}\sinh\alpha_1 x_{j+1} & \alpha_1^{2n}\cosh\alpha_1 x_{j+1} & \cdots & \alpha_n^{2n}\sinh\alpha_n x_{j+1} & \alpha_n^{2n}\cosh\alpha_n x_{j+1} \end{bmatrix} \mathbf{c} = \mathbf{u}, \tag{2.3}$$

where $\mathbf{c} = [c_0{}^j, c_1{}^j, a_1{}^j, b_1{}^j, \ldots, a_n{}^j, b_n{}^j]^T$ and $\mathbf{u} = [y_j, y_{j+1}, M_j{}^1, M_{j+1}^1, \ldots, M_j{}^n, M_{j+1}^n]^T$.

Dividing the matrix in (2.3) into 2×2 block submatrices, it is easy to see it is of the form

$$\begin{bmatrix} (A_0) & (A_1) & \cdots & (A_n) \\ (0) & \alpha_1{}^2(A_1) & \cdots & \alpha_n{}^2(A_n) \\ (0) & \alpha_1{}^4(A_1) & \cdots & \alpha_n{}^4(A_n) \\ \vdots & & & \vdots \\ (0) & \alpha_1^{2n}(A_1) & \cdots & \alpha_n^{2n}(A_n) \end{bmatrix}, \tag{2.4}$$

where

$$(A_0) = \begin{bmatrix} 1 & x_j \\ 1 & x_{j+1} \end{bmatrix}, \qquad (A_i) = \begin{bmatrix} \sinh \alpha_i x_j & \cosh \alpha_i x_j \\ \sinh \alpha_i x_{j+1} & \cosh \alpha_i x_{j+1} \end{bmatrix}, \qquad i = 1, \ldots, n,$$

and

$$(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrix (2.4) can be factored as

$$\begin{bmatrix} (\mathrm{I}) & (\mathrm{I}) & \cdots & (\mathrm{I}) \\ (0) & \alpha_1^2(\mathrm{I}) & \cdots & \alpha_n^2(\mathrm{I}) \\ \vdots & \vdots & & \vdots \\ (0) & \alpha_1^{2n}(\mathrm{I}) & \cdots & \alpha_n^{2n}(\mathrm{I}) \end{bmatrix} \begin{bmatrix} (A_0) & (0) & \cdots & (0) \\ (0) & (A_1) & \cdots & (0) \\ \vdots & & & \vdots \\ (0) & (0) & \cdots & (A_n) \end{bmatrix}, \tag{2.5}$$

where

$$(\mathrm{I}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, the matrix on the right in (2.5) is invertible iff $A_0, A_1, \ldots, A_n$ are. Since $x_{j+1} > x_j$, A_0 is invertible, and since $\det(A_i) = \cosh \alpha_i x_{j+1} \sinh \alpha_i x_j - \sinh \alpha_i x_{j+1} \cosh \alpha_i x_j = -\sinh \alpha_i l \neq 0$, $A_1, \ldots, A_n$ are invertible. Thus, we have reduced the question of the invertibility of the matrix in (2.3) to the invertibility of the left,matrix of (2.5). We have

THEOREM 2.1. *The generalized Vandermonde determinant*

$$\begin{vmatrix} (\mathrm{I}) & (\mathrm{I}) & \cdots & (\mathrm{I}) \\ \lambda_0(\mathrm{I}) & \lambda_1(\mathrm{I}) & \cdots & \lambda_n(\mathrm{I}) \\ \vdots & & & \vdots \\ \lambda_0^n(\mathrm{I}) & \lambda_1^n(\mathrm{I}) & \cdots & \lambda_n^n(\mathrm{I}) \end{vmatrix} = \left\{ \prod_{i>j} (\lambda_i - \lambda_j) \right\}^2. \tag{2.6}$$

The proof of this is an easy generalization of the standard proof for Vandermonde determinants.

By our assumption $0 < \alpha_1 < \cdots < \alpha_n$, we have that $0, \alpha_1^2, \ldots, \alpha_n^2$ are distinct and, therefore, it follows directly from Theorem 2.1 that the matrix on the left in (2.5) is invertible. It is this fact which ensures the validity of the representation of S_Δ in terms of y_j, y_{j+1}, $\{M_j^i, M_{j+1}^i\}_{i=1}^n$ for any operator of the form (2.1), and is the basis for this entire algebraic approach to generalized splines through the use of multiple unknowns.

To find the M_j^i's and thus specify the spline completely, we attempt to solve the system of equations resulting from the continuity conditions on $S_\Delta', S_\Delta''', \ldots, S_\Delta^{(2n-1)}$ at each x_j. This system is

$$\begin{aligned} C_1^1\mathbf{m}^1 + C_1^2\mathbf{m}^2 + \cdots + C_1^n\mathbf{m}^n &= \mathbf{a} \\ C_2^1\mathbf{m}^1 + C_2^2\mathbf{m}^2 + \cdots + C_2^n\mathbf{m}^n &= \mathbf{0} \\ \vdots \qquad\qquad\qquad\qquad & \vdots \quad \vdots \\ C_n^1\mathbf{m}^1 + C_n^2\mathbf{m}^2 + \cdots + C_n^n\mathbf{m}^n &= \mathbf{0} \end{aligned} \tag{2.7}$$

where $\mathbf{m}^i = [\ldots, M_{-1}^i, M_0^i, M_1^i, \ldots]^{\mathrm{T}}$, $i = 1, \ldots, n$, $\mathbf{a}$ is as in Section 1 and each C_j^i is a tri-diagonal, symmetric, doubly infinite Toeplitz matrix.

For example, septic splines, which are useful in applications, arise from the operator $L = D^4$, and although (2.2) becomes a linear combination of $1, x, x^2, \ldots, x^7$, we still get the following system of equations for the unknowns $M_j^1 = S''_\Delta(x_j)$, $M_j^2 = S_\Delta^{(iv)}(x_j)$, $M_j^3 = S_\Delta^{(vi)}(x_j)$, which we use to represent S_Δ

$$\begin{aligned}
&(1/3!)[\cdots 0\ 1\ 4\ 1\ 0 \cdots]\,\mathbf{m}^1 - (l^2/3 \times 5!)[\cdots 0\ 7\ 16\ 7\ 0 \cdots]\,\mathbf{m}^2 \\
&\qquad + (l^2/3 \times 7!)(\cdots 0\ 31\ 64\ 31\ 0 \cdots]\,\mathbf{m}^3 = \mathbf{a}, \\
&[\cdots 0\ -1\ 2\ -1\ 0 \cdots]\,\mathbf{m}^1 - (l^2/3!)[\cdots 0\ 1\ 4\ 1\ 0 \cdots]\,\mathbf{m}^2 \\
&\qquad + (l^4/3 \times 5!)[\cdots 0\ 7\ 16\ 7\ 0 \cdots]\,\mathbf{m}^3 = \mathbf{0}, \\
&[\cdots 0\ -1\ 2\ -1\ 0 \cdots]\,\mathbf{m}^2 + (l^2/3!)[\cdots 0\ 1\ 4\ 1\ 0 \cdots]\,\mathbf{m}^3 = \mathbf{0}.
\end{aligned} \tag{2.8}$$

As before, the procedure for solving (2.8) is substitution. We solve the last equation for $\mathbf{m}^3$ in terms of $\mathbf{m}^2$ and substitute the result into the second equation, which we then solve for $\mathbf{m}^2$ in terms of $\mathbf{m}^1$ and use both of these in the first equation. For polynomial splines, the first two equations always involve all the unknowns and succeeding equations involve one fewer unknown at each step until the last equation involves only the two unknowns $\mathbf{m}^n$, $\mathbf{m}^{n-1}$. For multiple hyperbolic splines this disappearance from succeeding equations does not occur; the substitution procedure is still carried out, however.

It is instructive, at this point, to mention, at least briefly, the technique of Schoenberg for calculating polynomial splines on R^1. He writes [4] a polynomial spline (of odd degree $k - 1$) on the uniform mesh $\{0, \pm 1, \pm 2, \ldots\}$ in the form

$$S_\Delta(x) = \sum_{j=-\infty}^{\infty} c_j Q_k(x - j),$$

where

$$Q_k(x) = (1/(k-1)!) \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x - i)_+^{k-1}$$

and

$$\begin{aligned} x_+^{k-1} &= x^{k-1} \qquad \text{if} \quad x \geqslant 0, \\ &= 0 \qquad\quad\ \ \text{if} \quad x < 0. \end{aligned}$$

Q_k vanishes outsides $(0, k)$, so the sum in the expression for S_Δ creates no problems.

The calculation of the coefficients c_j for given interpolation data $\{y_j\}$ involves a process identical to that of inverting the Toeplitz matrices which arise by our technique. To evaluate the spline explicitly, it is necessary to calculate the function $Q_k(x) = M_k(x - k/2)$ (see [4]). The major disadvantages to this approach are its apparent confinement to polynomial splines (it is not at all clear to this author, what, if anything, corresponding to Q_k, can be constructed for generalized splines; although let it not be assumed that Schoenberg makes any such claims for its extensibility) and the restriction to interpolation data satisfying $\| \Delta^k y \|_p < \infty$ where $\| \cdot \|_p$ is the usual norm for l_p $(1 \leqslant p \leqslant \infty)$,

$$\| x \|_p = \left(\sum_{j=-\infty}^{\infty} | x_j |^p \right)^{1/p} \qquad \text{if} \quad 1 \leqslant p \leqslant \infty,$$

and

$$\| x \|_\infty = \sup_j | x_j |.$$

The applicability of our approach to other types of constant coefficient differential operators L, such as those with irreducible quadratic factors (over the reals) or with repeated factors, is expected to be much the same.

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